REDUNDANT POLES AND S-MATRIX FOR GENERAL POTENTIAL

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ABSTRACT

The purpose of this paper is to show that at "redundant poles" the linearindependence of two basic solutions used for evaluating usual*S*-matrix for a general potential

$$V(r) = -\frac{n(n+1)\alpha^2}{\cosh^2 \alpha r},$$

breaks down.

Keywords: General potential, redundant poles, *S*-matrix, linear independence of solutions

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1. INTRODUCTION

Several authors [1-6], applying suitable functional transformations to a second order differential equation, have constructed solvable potentials for the non-relativistic Schrödinger equation, the relativistic Klein-Gordon and Dirac equations. The potentials were obtained from hypergeometric, confluent hypergeometric and Bessel differential equations. A potential of the type

$$V(r) = -\frac{n(n+1)\alpha^2}{\cosh^2 \alpha r} \quad (1)$$

was constructed by Sharma [7] by transforming the associated Legendre differential equation following the method used in our previous papers. This potential derives its importance from the fact that for particular values on n = 1 it gets reduced to the potential already derived [8]. Similarly for n = 1 and replacing $\alpha$by $-\alpha/2$, it takes the form of an Eckart [9] type of potential (with special value β = 1).

It is well-known that the poles of the s-matrix in the upper half plane of the complex momentum variable correspond to genuine bound states of the system; and a set of states must include these bound states before they constitute a complete set. It was shown [10] that in the case of an exponential potential for the s- wave, S – matrix, there exist poles that do not contribute to the completeness even though they appear in the same part of k- plane as the bound state poles. These poles have been referred to as "redundant poles" in the literature and their importance with regard to the concept of "shadow states" has been stressed by few authors [11]. In particular, theseauthors have depicted by specific examples that it is possible to have two phase-equivalent systems in which complete sets of states of one potential include only the scattering states, while for the other potential there is a bound state that must be included. Specific cases of exponential and Eckart potentials were reexamined [12]and it has been shown that "redundant" solutions for these potentials correspond to the vanishing of the Wronskian pointing to the breakdown of the linear independence of the starting wave-functions employed in deriving the expression for the S- matrix.

The object of this communication is to show that for the case of general potential (1), considered in this note also, the "redundant" poles correspond to the vanishing of the relevant Wronskian showing thereby that their linear independence breaks down at such poles.

2. REGULAR SOLUTIONS AND THE *S*-MATRIX

As only the regular solutions are acceptable for the discussion of the physical problems, the regular solution [13] for the potential (1) defined by the boundary condition $\phi(r)$ = 0 for r = 0 is written as

$$\phi(r) = X\, 2^{-\frac{ik}{\alpha}} e^{ikr}\, {}_2F_1\left(\frac{n+1}{2}+\frac{ik}{2\alpha}, -\frac{n}{2}+\frac{ik}{2\alpha}; 1+\frac{ik}{\alpha}; 1\right) \times {}_2F_1\left(\frac{n+1}{2}-\frac{ik}{2\alpha}, -\frac{n}{2}-\frac{ik}{2\alpha}; 1-\frac{ik}{\alpha}; \operatorname{sech}^2 \alpha r\right) -$$
$$Y\, 2^{\frac{ik}{\alpha}} e^{-ikr}\, {}_2F_1\left(\frac{n+1}{2}-\frac{ik}{2\alpha}, -\frac{n}{2}+\frac{ik}{2\alpha}, 1+\frac{ik}{\alpha}^2; \operatorname{sech}^2 \alpha r\right).$$

(2)

Thus the S-matrix will have the form

$$S(k) = \frac{2^{\frac{2ik}{\alpha}}\, {}_2F_1\left(\frac{1+n}{2}-\frac{ik}{2\alpha}, -\frac{n}{2}-\frac{ik}{2\alpha}; 1-\frac{ik}{\alpha}; 1\right)}{{}_2F_1\left(\frac{1+n}{2}+\frac{ik}{2\alpha}, \frac{n}{2}+\frac{ik}{2\alpha}; 1+\frac{ik}{\alpha}; 1\right)} \qquad (3)$$

On further simplifying [13], equation (3) can be written as

$$S(k) = 2^{\frac{2ik}{\alpha}} \frac{\Gamma\left(1-\frac{ik}{\alpha}\right)}{\Gamma\left(1+\frac{ik}{\alpha}\right)} \frac{\Gamma\left(\frac{1-n}{2}+\frac{ik}{2\alpha}\right)}{\Gamma\left(\frac{1-n}{2}-\frac{ik}{2\alpha}\right)} \frac{\Gamma\left(1+\frac{n}{2}+\frac{ik}{2\alpha}\right)}{\Gamma\left(1+\frac{n}{2}-\frac{ik}{2\alpha}\right)} \qquad (4)$$

Applying Legendre's duplication formula [13], the S-matrix can be finally written in the following form

$$S(k) = \Pi \frac{k-in\alpha}{k+in\alpha} \qquad (5)$$

The unitarity of S-matrix [14] can also be established from equation (5).

Now the poles of the$S$-matrix obviously are given by such values of $k$which satisfy the relation

$k = -in\alpha,$ n, being an integer $(\neq 0)$ (6)

From the regular solution (2), the Wronskian of

$${}_2F_1\left(\frac{n+1}{2}-\frac{ik}{2\alpha}, -\frac{n}{2}-\frac{ik}{2\alpha}; 1-\frac{ik}{\alpha}; \operatorname{sech}^2 \alpha r\right)$$

and

$${}_2F_1\left(\frac{n+1}{2}+\frac{ik}{2\alpha}, \frac{-n}{2}+\frac{ik}{2\alpha}; 1+\frac{ik}{\alpha}; \operatorname{sech}^2 \alpha r\right)$$

which are two linearly independent solutions (without considering other factors) is given as [13]

$$W\begin{bmatrix} {}_2F_1\left(\frac{n+1}{2}-\frac{ik}{2\alpha},\frac{-n}{2}-\frac{ik}{2\alpha};1-\frac{ik}{2\alpha};\text{sech}^2\alpha r\right), \\ {}_2F_1\left(\frac{n+1}{2}+\frac{ik}{2\alpha},\frac{-n}{2}+\frac{ik}{2\alpha};1+\frac{ik}{\alpha};\text{sech}^2\alpha r\right) \end{bmatrix}=$$

$$\frac{\left(\frac{n+1}{2}+\frac{ik}{2\alpha}\right)\left(\frac{-n}{2}+\frac{ik}{2\alpha}\right)}{\left(1+\frac{ik}{\alpha}\right)}(2\alpha\,\text{sech}^2\alpha r\tanh\alpha r)$$

$$\times\{{}_2F_1\left(\frac{n+1}{2}-\frac{ik}{2\alpha},\frac{-n}{2}-\frac{ik}{2\alpha};1-\frac{ik}{\alpha};\text{sech}^2\alpha r\right)$$

$$\times{}_2F_1\left(\frac{n+3}{2}+\frac{ik}{2\alpha},1-\frac{n}{2}+\frac{ik}{2\alpha};2+\frac{ik}{\alpha};\text{sech}^2\alpha r\right)\}$$

$$-\frac{\left(\frac{n+1}{2}-\frac{ik}{2\alpha}\right)\left(\frac{-n}{2}-\frac{ik}{2\alpha}\right)}{\left(1-\frac{ik}{\alpha}\right)}(2\alpha\,\text{sech}^2\alpha r\tanh\alpha r)$$

$$=\begin{Bmatrix} {}_2F_1\left(\frac{n+1}{2}+\frac{ik}{2\alpha},\frac{-n}{2}+\frac{ik}{2\alpha};1+\frac{ik}{\alpha}\text{sech}^2\alpha r\right) \\ \times{}_2F_1\left(\frac{n+3}{2}-\frac{ik}{2\alpha}.1-\frac{n}{2}-\frac{ik}{2\alpha};2-\frac{ik}{\alpha};\text{sech}^2\alpha r\right) \end{Bmatrix} \qquad (7)$$

Now it is easy to verify that the Wronskian vanishes for the values of $k$given by (6), showing thereby that the starting solutions (2) which are linearly independent, are not so at such values of$k$.

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